International Journal of Theoretical Physics, Vol. 31, No. 7, 1992

Higher-Order Mechanical Systems with Constraints

Manuel de León[1] and Paulo R. Rodrigues[2]

Received August 22, 1991

We extend Weber's approach for mechanical systems with constraints in terms of differential-geometric structures underlying higher-order tangent bundles.

1. INTRODUCTION

Differential geometry provides a good framework for studying Lagrangian and Hamiltonian formalisms of classical mechanics. However, something lacking on constraints and their geometric meaning has been remarked [see the final comments in Spivak (1979), Vol. V, p. 600, for instance]. Constrained Lagrangian systems are studied in Arnold (1988) and a particular case is presented (called *vakanomic mechanics*) due to Koslov (1983). This kind of mechanics is based on a natural generalization of Hamilton's principle [see Arnold (1988) for further details]. Constrained systems were also considered by Dirac (1964), but with a different meaning, since they are characterized by degenerate Lagrangians.

More recently, Weber (1985) developed a geometric formulation of Hamiltonian systems with constraints using a set $C=\{\alpha_1, \ldots, \alpha_r\}$ of linearly independent 1-forms on a symplectic manifold (S, ω) of dimension $2n$, $r \leq n$. If this set defines an integrable distribution on S, then C is said to be *holonomic* (otherwise it is said to be *nonholonomic* or *anholonomic*). If we suppose that S is fibered over an n-dimensional manifold M, then holonomic and nonholonomic constraints can be classified in *basic* or *semibasic* 1-forms with respect to this fibration. Under certain conditions it is possible to show that holonomic (resp. nonholonomic) constraints can be canonically

[1]Unidad de Matemáticas, Consejo Superior de Investigaciones Científicas, Serrano 123, 28006 Madrid, Spain.
[2]Departamento de Geometria, Instituto de Matematica, Universidade Federal Fluminense, 24020-Niteroi, RJ, Brazil.

0020-7748/92/0700-1303$06.50/0

transformed into basic (resp. basic or semibasic) 1-forms locally expressed by dx^A [resp. $\alpha^a_A(x)\,dx^A$ or $\alpha^a_A(x,y)\,dx^A$], $1 \leq A \leq n$, $1 \leq a \leq r$, where (x^A) [resp. (x^A, y^A)] are local coordinates for M (resp. S).

With such a kind of geometric classification Weber examines regular Hamiltonian (and Lagrangian) systems with constraints. Taking $S = T^*M$ as the cotangent bundle of the configuration manifold M and a function $H: T^*M \to R$ as the Hamiltonian, we may consider a set of linearly independent 1-forms $C = \{\alpha_1, \ldots, \alpha_r\}$, $r \leq n$, on T^*M. The corresponding motion equations are

$$i(X)\omega = dH + \Lambda^a \alpha_a, \qquad \alpha_a(X) = 0$$

where ω is the canonical symplectic form on T^*M and the Λ's are Lagrange multipliers. If we assume that H is regular [the matrix $(\partial^2 H/\partial p^A\,\partial p^B)$ is invertible everywhere, (q^A, p^A) are local coordinates for T^*M], then a transformation $Ham: T^*M \to TM$ is used to obtain the Lagrangian counterpart. We have a new system (TM, ω_L, E), where TM is the tangent bundle of M, ω_L is the symplectic form on TM, obtained from ω via the inverse of Ham, denoted by Leg, and E is the energy of the new system, defined by $E \circ Ham = H$. The motion equations are given in the symplectic form

$$i(Y_E)\omega_L = dE$$

but now with a significant difference: Y_E must be a second-order differential equation, since the integral curves of Y_E must satisfy the Euler–Lagrange equations. A condition for this is that the energy E be locally defined by

$$E = v^A(\partial L/\partial v^A) - L$$

where $L: TM \to R$ is the Lagrangian and (q^A, v^A) are coordinates for TM. In such a case Weber says that (TM, ω_L, E) is a (regular) *mechanical system*. The constrained situation is now obtained by pulling back the original constraints on T^*M to TM via $Leg: TM \to T^*M$ and Weber proves that *a (regular) Hamiltonian system with constraints admits a mechanical system if and only if the constraints are of semibasic (or basic) type.*

Our purpose is to extend Weber's viewpoint to mechanical systems of higher order, i.e., systems which are not only dependent on position–velocity coordinates, but also accelerations and higher-order derivatives with respect to the time [for a local description see Whittaker (1959), p. 265]. First we derive a global formulation on higher-order tangent bundles (a natural generalization of tangent bundles) for Lagrangians with constraints, *without assuming the existence of a Hamiltonian counterpart*. Second we examine the Hamiltonian case. In order to do so, a higher-order *almost tangent geometry* is used (Clark and Bruckheimer, 1960; Eliopoulos, 1962). Klein (1962) showed that this kind of geometric structure plays a role in Lagrangian dynamics like symplectic geometry in Hamiltonian dynamics [further details about almost tangent geometry can be found in Godbillon (1969) or de León and Rodrigues (1989)].

Finally, we would like to remark that this work was in its final form when the authors noticed a recent paper by Cardin and Zanzotto (1989) on (first order) mechanical systems with (holonomic) constraints. For instance, in the definition of a constrained mechanical system one requires an embedded constraint manifold Q with no boundary in the configuration manifold and an appropriate subbundle of the vector bundle of the semibasic forms restricted to the tangent bundle of Q.

This assumption is not adopted here since the (holonomic) system of constraints C defines a $(2n-r)$-dimensional distribution D on S $(=TM$ or $T^*M)$

$$D(x)=\{X\in T_xS/\alpha_a(X)=0,\ 1\leq a\leq r\}$$

for each $x\in S$. Hence C is holonomic if the ideal Ξ of ΛS generated by C satisfies $d\Xi\subset\Xi$, i.e., Ξ is a differential ideal. Thus, for a system of holonomic constraints the motion lies on a specific leaf of the foliation defined by D, that is, the constraints emerge as *foliations* of the phase space S.

The paper is divided in the following way. In Section 2 we recall some results about the higher-order formalism and some essential terminology [de León and Rodrigues (1985) give further details; see also the Appendix], which is still nonstandard in the literature. In Section 3 we give a *different* insight about extending Weber's results to higher-order Lagrangian systems with constraints; the Hamiltonian version is also presented. An example is given in Section 4.

2. BACKGROUND

Let T^sQ be the tangent bundle of order s of an m-dimensional manifold Q, i.e., T^sQ is the manifold of s-jets at $0\in R$ of curves in Q. If (q^A), $1\leq A\leq m$, are coordinates in Q, we denote by (z_i^A), $1\leq A\leq m$, $0\leq i\leq s$, the induced coordinates in T^sQ. If $\sigma(t)=(q^A(t))$, then

$$z_i^A(j_0^s\sigma)=(1/i!)(d^i/dt^i)(q^A(t))_{t=0}$$

We set $z_0^A=q^A$. Sometimes we use coordinates (q_i^A), where

$$q_i^A=(i!)z_i^A,\qquad 1\leq A\leq m,\qquad 0\leq i\leq s$$

The canonical projection $\rho_r^s\colon T^sQ\to T^rQ$, $r<s$, is defined by $\rho_r^s(j_0^s\sigma)=j_0^r\sigma$. We denote by J_1 the *canonical almost tangent structure of order* s on T^sQ. It is a (1, 1)-tensor field such that locally

$$J_1=\sum_{i=1}^{s}(\partial/\partial z_i^A)\otimes(dz_{i-1}^A)\tag{1}$$

We set $J_u=(J_1)^u$ (i.e., J_1 u-times). From (1) we deduce that $J_u=0$ when $u>s+1$. By C_1 we denote the *Liouville vector field* on T^sQ, locally given by

$$C_1=\sum_{i=1}^{s} (i)z_i^A(\partial/\partial z_i^A)$$

We set $C_u=J_{u-1}C_1$, $2\leq u\leq s$. It can be shown that a vector field ξ on T^sQ is an $(s+1)$*th-order differential equation* if and only if $J_1\xi=C_1$; sometimes $(s+1)$th-order differential equations will be called *semisprays of type 1* [*sprays* were introduced by Ambrose *et al.* (1960) for homogeneous second-order differential equations].

Semibasic forms are defined as a natural generalization of the standard case of first order: a 1-form α on T^sQ is said to be *semibasic of type u* if $\alpha\in\operatorname{Im} J_u^*$, where J_u^* is the adjoint of J_u acting on forms. Hence, α is semibasic of type u if and only if α is locally expressed by

$$\alpha=\sum_{i=0}^{s-u} \alpha_A^i(z^A, z_1^A, \ldots, z_s^A)\, dz_i^A \tag{2}$$

where the above sum is taken over $0\leq i\leq s-u$.

Let us consider now a Lagrangian function L of order k, that is, a function L: $T^kQ\to R$. There exists a 1-form α_L on $T^{2k-1}Q$ intrinsically defined by

$$\alpha_L=\sum_{i=0}^{k-1} \frac{1}{i}(-1)^i d_T^i d_{J_{i+1}}L$$

where $d_T^i=d_T\cdots d_T$ (i times) and d_T is the operator which maps each function f on T^sQ into a function d_Tf on $T^{s+1}Q$ locally given by

$$d_Tf(q_0^A, \ldots, q_s^A)=\sum_{i=0}^{s} q_{i+1}^A(\partial f/\partial q_i^A)$$

(d_T extends naturally to an operator mapping p-forms on T^sQ into p-forms on $T^{s+1}Q$ and we use the same symbol for both situations). Then the coordinate expression of α_L is

$$\alpha_L=\sum_{i=0}^{k-1} p_{i+1}^A\, dq_i^A$$

where

$$p_{i+1}^A=\sum_{j=0}^{k-i-1} (-1)^j d_T^j(\partial L/\partial q_{i+j+1}^A), \qquad 0\leq i\leq k-1 \tag{3}$$

Remark 2.1. Notice that p^A_{i+1} depends at most in q^A_{2k-1-i} and

$$\frac{\partial p^A_{i+1}}{\partial q^B_{2k-1-i}}=(-1)^{k-1-i}\left(\frac{\partial^2 L}{\partial q^A_k\,\partial q^B_k}\right)$$

Thus, α_L is a *semibasic 1-form of type k* on $T^{2k-1}Q$ and defines a mapping (see the Appendix)

$$Leg\colon\ \ T^{2k-1}Q\to T^*(T^{k-1}Q)$$

such that

$$\pi_{T^{k-1}Q}\circ Leg=\rho^{2k-1}_{k-1}\qquad\text{and}\qquad Leg^*(\lambda_{k-1})=\alpha_L$$

where λ_{k-1} is the canonical Liouville form on $T^*T^{k-1}Q$. In local coordinates we have

$$Leg(q^A_0,\ldots,q^A_{2k-1})=(q^A_0,\ldots,q^A_{k-1},p^A_1,\ldots,p^A_k)$$

Leg is called the *Legendre–Ostrogradskii* transformation. It is the natural generalization of the Legendre transformation to higher-order theories and, for the regular case, gives the way to pass from the Lagrangian to the Hamiltonian formulation. In fact, (3) may be taken as the definition of generalized momenta. If $\omega_{k-1}=-d\lambda_{k-1}$ is the canonical symplectic form on $T^*(T^{k-1}Q)$, we have

$$Leg^*\omega_{k-1}=\omega_L$$

where $\omega_L=-d\alpha_L$.

A Lagrangian L is said to be *regular* if the Hessian matrix $(\partial^2L/\partial q^A_k\,\partial q^B_k)$ is nonsingular. It can be shown that *L is regular iff ω_L is symplectic iff $Leg\colon T^{2k-1}Q\to T^*(T^{k-1}Q)$ is a local diffeomorphism*. For regular Lagrangians one defines intrinsically the *energy function* E_L on $T^{2k-1}Q$ by

$$E_L=\sum_{i=1}^{k}\frac{1}{i}(-1)^{i-1}d_T^{i-1}(C_iL)-L$$

and the vector field ξ_L defined by $i(\xi_L)\omega_L=dE_L$ is a $2k$th-order differential equation (or semispray of type 1) on $T^{2k-1}Q$. Furthermore, the solutions of ξ_L satisfy the *Euler–Lagrange equations* for L:

$$\sum_{i=0}^{k}(-1)^i(d^i/dt^i)(\partial L/\partial q^A_i)=0$$

Thus, if L is a regular Lagrangian, we call $(T^{2k-1}Q,\omega_L,E_L)$ a *regular Lagrangian system of order k*.

3. LAGRANGIAN AND HAMILTONIAN SYSTEMS OF HIGHER ORDER WITH CONSTRAINTS

Let $C=\{\theta_1, \ldots, \theta_r\}$ be a set of r linearly independent 1-forms on $T^{2k-1}Q$. Then $(T^{2k-1}Q, \omega_L, E_L, C)$ is called a *regular Lagrangian system with constraints*. Let us now consider the equation

$$i(\xi)\omega_L = dE_L + \sum_{a=1}^{r} \Lambda^a\theta_a, \qquad \theta_a(\xi)=0, \quad \Lambda^a \neq 0, \quad \forall a \tag{4}$$

Definition 3.1. We say that $(T^{2k-1}Q, \omega_L, E_L, C)$ represents a *mechanical system of order* k (with constraints) if the vector field ξ given by (4) is a $2k$th-order differential equation.

We recall that there always exists such ξ, since ω_L is symplectic, but ξ is not necessarily of $2k$th-order type.

Lemma 3.1. Let X be a vector field on $T^{2k-1}Q$ given by $i(X)\omega_L=\theta$. Then $J_1X=0$ if and only if θ is semibasic of type $2k-1$.

Proof. Suppose that $J_1X=0$. Then $X=X^A(\partial/\partial q^A_{2k-1})$. As

$$\omega_L = \sum_{i=0}^{k-1} dq_i^A \wedge dp_{i+1}^A$$

we have

$$\theta = -\left(X^A \frac{\partial L}{\partial q_k^A\, \partial q_k^B}\right) dq_0^B$$

Thus, θ is semibasic of type $2k-1$. Conversely, suppose that θ is semibasic of type $2k-1$. Then $\theta=\theta_A\, dq_0^A$. We set

$$X = \sum_{i=0}^{2k-1} X_i^A \frac{\partial}{\partial q_i^A}$$

Then, taking into account the remark of Section 2 and developing $i(X)\omega_L = \theta$ in local coordinates, one obtains, for instance,

$$X_0^A \left(\frac{\partial p_1^A}{\partial q_{2k-1}^B}\right) = 0$$

that is, $X_0^A=0$. Also, the same development shows that among the remaining terms

$$X_1^A = X_2^A = \cdots = X_{2k-2}^A = 0$$

[indeed, it is sufficient to see that $(i(X)\omega_L)(\partial/\partial q^A_{2k-1})=0$, etc]. Thus, $J_1X=0$. ■

Theorem 3.1. $(T^{2k-1}Q, \omega_L, E_L, C)$ represents a mechanical system of order k if and only if the 1-forms θ_a are semibasic of type $2k-1$.

Proof. Let X_a be the vector fields given by $i(X_a)\omega_L = \theta_a$, $1 \le a \le r$. Then from (4) one has that

$$i(\xi - \Lambda^a X_a)\omega_L = dE_L = i(\xi_L)\omega_L$$

i.e.,

$$\xi = \xi_L + \Lambda^a X_a$$

and ξ is a semispray of type 1 iff $J_1\xi = C_1$ iff $J_1 X_a = 0$, $1 \le a \le r$. But $J_1 X_a = 0$ iff θ_a is semibasic of type $2k-1$ by the above lemma. ■

If $(T^{2k-1}Q, \omega_L, E_L, C)$ represents a mechanical system of order k, then the 1-forms θ_a are locally given by

$$\theta_a = (\theta_a)_A(q_0^A, \ldots, q_{2k-1}^A)\, dq_0^A$$

This implies that $r \le m$. A direct computation shows that the solutions of (4) satisfy the following *Euler–Lagrange equations with constraints*:

$$\sum_{i=0}^{k} (-1)^i (d^i/dt^i)(\partial L/\partial q_i^A) = \Lambda^a(\theta_a)_A, \qquad 1 \le A \le m$$

In addition, since $\theta_a(\xi) = 0$, we have $(\theta_a)_A(\dot{q}^A) = 0$, $1 \le a \le r$. Thus, a direct constraint formulation for Lagrangian systems was performed without using first the constraint formulation for Hamiltonian systems as proposed by Weber.

If we want to examine the Hamiltonian counterpart of such a kind of Lagrangian theory, then we first examine the above Legendre–Ostrogradskii transformation. If *Leg* is only a local diffeomorphism, then we define the local Hamiltonian function H on $T^*(T^{k-1}Q)$ by $H \circ Leg = E_L$. If *Leg* is a global diffeomorphism, i.e., L is hyperregular, then we have the equivalence of the both formulations and the Hamiltonian function H is globally defined.

Let us examine the other direction, that is, we suppose that we are *starting* with a Hamiltonian system defining a mechanical system of higher order. Consider a regular Hamiltonian system $(T^*(T^{k-1}Q), \omega_{k-1}, H, C)$ with constraints $C = \{\theta_1, \ldots, \theta_r\}$. We denote by

$$(q_0^A, \ldots, q_{k-1}^A, p_1^A, \ldots, p_k^A)$$

the induced coordinates in $T^*(T^{k-1}Q)$. If H is regular [the Hessian matrix $(\partial^2 H/\partial p_i^A\, \partial p_j^B)$, $1 \le i, j \le k$, $1 \le A$, $B \le m = \dim Q$, is nonsingular everywhere], then there exists a fiber-preserving mapping *Ham*:

$$T^*(T^{k-1}Q) \to T(T^{k-1}Q)$$

locally defined by

$$Ham(q_i^A, p_i^A) = (q_i^A, (\partial H/\partial p_i^A))$$

and *Ham* is a local diffeomorphism. Thus, we may define a function L': $T(T^{k-1}Q) \to R$ given by $L'(q_i^A, \dot{q}_i^A) = \sum_{i=1}^{k} (\partial H/\partial p_i^A)p_i^A - H$.

Now, let j: $T^kQ \to T(T^{k-1}Q)$ be a canonical injection locally defined by

$$j(q_0^A, \ldots, q_k^A) = (q_0^A, \ldots, q_{k-1}^A; q_1^A, \ldots, q_k^A)$$

A Lagrangian function L: $T^kQ \to R$ of order k may be defined by $L = L' \circ j$. A direct computation shows that L is regular and the corresponding Legendre transformation Leg: $T^{2k-1}Q \to T^*(T^{k-1}Q)$ satisfies $H \circ Leg = E_L$.

Definition 3.2. $(T^*(T^{k-1}Q), \omega_{k-1}, H, C)$ *defines a mechanical system of order* k if $(T^{2k-1}, \omega_L, E_L, C^* = Leg^*C)$ represents a mechanical system of order k.

It is not hard to show the following result.

Theorem 3.2. $(T^*(T^{k-1}Q), \omega_{k-1}, H, C)$ defines a mechanical system of order k iff all the 1-forms θ_a are semibasic with respect to the fibration

$$\beta^{k-1} \circ \pi_{T^{k-1}Q}: \quad T^*(T^{k-1}Q) \to Q$$

(i.e., the θ_a vanishes along the $\beta^{k-1} \circ \pi_{T^{k-1}Q}$-vertical vector fields, for all a) where β^{k-1}: $T^{k-1}Q \to Q$, resp., $\pi_{T^{k-1}Q}T^*(T^{k-1}Q) \to T^{k-1}Q$, are the canonical projections.

Remark 3.3. Let (S, ω) be a symplectic manifold. Two 1-forms α and β are said to be in *involution* if the bracket $\{\alpha, \beta\} = \omega(X_\alpha, X_\beta)$ vanishes, where X_α (resp. X_β) is the vector field on S defined by $i_{X_\alpha}\omega = \alpha$ (resp. $i_{X_\beta}\omega = \beta$). For the *holonomic case* it is always possible to show that the 1-forms $\{\alpha_1, \ldots, \alpha_r\}$ are in involution iff there is a local symplectomorphism such that the original 1-forms α_a, $1 \leq a \leq r$, are transformed into semibasic forms [for a proof see Jacobi's theorem in Duistermaat (1973), p. 100, and the Lie corollary in Abraham and Marsden (1978), p. 419]. This result is valid, under certain circumstances, for the nonholonomic situation, as was shown by Weber.

4. AN EXAMPLE

Consider an elastic beam in the Euclidean space R^3 and suppose that the x axis coincides with the beam. If no external forces act on the beam, then its equilibrium configuration is a straight line. But external forces acting on the beam induce deflections. These deflections can be represented by points of a plane M perpendicular to the beam, i.e., M is the yz plane. In the sequel we put $q^1 = y$, $q^2 = z$.

The dynamics of the beam is given by a Lagrangian function of order 2,

$$L(q, \dot{q}, \ddot{q}) = \tfrac{1}{2} k_{ij} \ddot{q}^i \ddot{q}^j$$

$1 \le i, j \le 2$, where (k_{ij}) is a nonsingular symmetric matrix which characterizes the elastic properties of the beam (Abraham and Marsden, 1978, pp. 488, 489). Thus,

$$(\partial^2 L / \partial \ddot{q}^i \, \partial \ddot{q}^j) = (k_{ij})$$

and then $L: T^2 M \to R$ is a regular Lagrangian.

The momenta p_1^i and p_2^i are defined by

$$p_1^i = \frac{\partial L}{\partial \dot{q}^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}^i}\right) = -\sum_{j=1}^{2} k_{ij} \dddot{q}^j \tag{5}$$

$$p_2^i = \partial L / \partial \ddot{q}^i = \sum_{j=1}^{2} k_{ij} \ddot{q}^j, \qquad 1 \le i \le 2 \tag{6}$$

and the Euler–Lagrange equations are

$$\sum_{j=1}^{2} k_{ij} \ddddot{q}^j = 0, \qquad 1 \le i \le 2 \tag{7}$$

[We remark that (5)–(7) can be rewritten as

$$p_1^i + \dot{p}_2^i = 0, \qquad p_2^i = k_{ij} \ddot{q}^j, \qquad \dot{p}_1^i = 0$$

which are the equilibrium conditions of the beam (Abraham and Marsden, 1978, p. 489).]

The configuration of the bent beam is a differentiable curve in R^3 and $(q^1(t), q^2(t)) = (y(t), z(t))$ is its projection onto M. We impose the constraints $\theta_1 = dq^1$. This geometrically means that the bent beam is a curve which lies in the xz plane. Thus, $(q^1(t), q^2(t))$ satisfies the Euler–Lagrange equations with constraints

$$k_{1j} \frac{d^4 q^j}{dt^4} = \lambda, \qquad k_{2j} \frac{d^4 q^j}{dt^4} = 0$$

In addition, we have

$$\theta_1\left(\frac{dq^1}{dt}\frac{\partial}{\partial q^1}+\frac{dq^2}{dt}\frac{\partial}{\partial q^2}\right)=0$$

which implies

$$\dot{q}^1=0$$

Hence

$$k_{12}\frac{d^4q^2}{dt^4}=\lambda, \qquad k_{22}\frac{d^4q^2}{dt^4}=0$$

which leads to

$$\frac{d^4q^2}{dt^4}=0 \qquad \left(\text{or } \frac{d^4z}{dt^4}=0\right)$$

(see Kármán and Biot, 1940, p. 269).

APPENDIX

Let α be a semibasic form of type u on T^sQ. Then α defines a mapping

$$D_\alpha: \quad T^sQ \to T^*(T^{s-u}Q)$$

given by

$$\langle X, D_\alpha(j_0^s\sigma)\rangle = \alpha(j_0^{s-u}\sigma)(\bar{X})$$

where

$$X \in T_{j_0^{s-u}\sigma}(T^{s-u}Q) \qquad \text{and} \qquad \bar{X} \in T_{j_0^s\sigma}(T^sQ)$$

and such that

$$T\rho^s_{s-u}(\bar{X})=X$$

Since α is semibasic, D_α is well-defined. If α is locally given by (2), then we obtain

$$D_\alpha(z_0^A, z_1^A, \ldots, z_s^A)=(z_0^A, \ldots, z_{s-u}^A, \sigma_A^0, \ldots, \alpha_A^{s-u})$$

From which we easily deduce the following properties:

1. The diagram

$$\begin{tikzcd} T^sQ \arrow[rr] \arrow[dr, "\rho^s_{s-u}"'] & & T^*(T^{s-u}Q) \arrow[dl, "\pi_{T^{s-u}Q}"] \\ & T^{s-u}Q & \end{tikzcd}$$

is commutative, where $\pi_{T^{s-u}Q}$ is the canonical projection.

2. $D^*_\alpha \lambda_{s-u} = \alpha$, where λ_{s-u} is the Liouville form on $T^*(T^{s-u}Q)$.

ACKNOWLEDGMENTS

M.d.L. was supported in part by DGICYT-SPAIN, Proyecto PB88-0012, and FAPERJ-BRAZIL Proc. Proc. E-29/170.454/89. P.R.R. was supported in part by CNPq-BRAZIL, Proc. 30.1115/79.

REFERENCES

Abraham, R., and Marsden, J. (1978). *Foundations of Mechanics*, 2nd ed., Benjamin, New York.

Ambrose, W., Palais, R., and Singer, W. (1960). *Anais Academia Brasileira de Ciencas*, **32**, 163–178.

Arnold, V. I., ed. (1988). *Dynamical Systems III*, Encyclopaedia of Mathematical Sciences, Springer, Berlin.

Cardin, F., and Zanzotto, G. (1989). *Journal of Mathematical Physics*, **30**, 1473–1479.

Clark, R., and Bruckheimer, M. (1960). *Comptes Rendus de l'Academie des Sciences Paris*, **251**, 627–629.

de León, M., and Rodrigues, P. R. (1985). *Generalized Classical Mechanics and Field Theory*, North-Holland, Amsterdam.

de León, M., and Rodrigues, P. R. (1989). *Methods of Differential Geometry in Analytical Mechanics*, North-Holland, Amsterdam.

Dirac, P. A. M. (1964). *Lectures on Quantum Mechanics*, Belfer Graduate School of Science, New York.

Duistermaat, J. (1973). *Fourier Integral Operators*, Courant Institute of Mathematical Sciences, New York.

Eliopoulos, H. (1962). *Comptes Rendus de l'Academie des Sciences Paris*, **255**, 1563–1565.

Godbillon, C. (1969). *Géométrie Différentielle et Mécanique Analytique*, Hermann, Paris.

Kármán, Th., and Biot, M. A. (1940). *Mathematical Methods in Engineering*, McGraw-Hill, New York.

Klein, J. (1962). *Annales de l'Institut Fourier* (Grenoble) **12**, 1–124.

Kozlov, V. V. (1983). *Soviet Physics Doklady*, **28**, 735–737.

Spivak, M. (1979). *A Comprehensive Introduction to Differential Geometry*, Vol. 5, Publish or Perish, Berkeley, California.

Tulczyjew, W. (1975). *Comptes Rendus de l'Academie des Sciences Paris A*, **280**, 1295–1298.

Weber, R. W. (1985). *Archive for Rational Mechanics and Analysis*, **91**, 309–335.

Whittaker, E. T. (1959). *A Treatise on the Analytical Dynamics of Particle and Rigid Bodies*, Cambridge University Press, Cambridge.